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On a series expansion for the solitary wave

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The first 27 terms in a series expansion for the profile of a solitary wave are computed. From this, series expansions for the wave amplitude, mass and potential energy are obtained. A previous study indicated that the partial sums of these series converged for small- to medium-amplitude waves and that the diagonal Padé approximants converged for waves of all amplitudes. The data derived here show that this is not the case and that apparent convergence of Padé approximants can be misleading.

1. Introduction

In recent years, several methods have been used to study the properties of large-amplitude solitary waves. The wave of maximum amplitude, in particular, has been the focus of much attention. Longuet-Higgins & Fenton (1974) computed the values of several parameters for solitary waves of all amplitudes (up to the maximum) by means of partial power series and Padé approximants. Witting's (1981) study was based on a Fourier-series technique, with a singular term giving the sharp crest of the highest wave. Williams (1981) investigated periodic waves of maximum height by including two terms designed to produce sharp crests of the proper form. His large-wavelength data may be applied to solitary waves. Hunter & Vanden-Broeck (1983) extended the work of Lenau (1966), which also incorporates a singularity. The data of Pennell & Su (1984, hereinafter referred to as I), which are based on a 17-term series expansion, agree with those of Longuet-Higgins & Fenton but differ from those in the other aforementioned works.

In this paper we obtain the first 27 terms in a series expansion for the solitary wave. Here, however, we do not observe the convergence of the Padé approximants for the momentum and potential energy of very high waves that was indicated in I. The Padé approximants for the amplitude of the highest wave do appear to converge, although not to the value found by Williams, Witting and Hunter & Vanden-Broeck. If the latter value is assumed to be correct, as seems extremely probable given the excellent agreement among the three methods, then we may draw two conclusions: values obtained by Padé approximants based on only a 27-term series expansion are incorrect; and a sequence of Padé approximants may diverge even when it appears to be converging.

2. Formulation of the problem

We consider the two-dimensional motion of a solitary wave on the surface of an inviscid fluid in an open channel with a horizontal bottom. We make the flow steady by using a frame of reference which moves with the wave speed c. Furthermore, we assume that the flow is irrotational and that the fluid has uniform density. Let the x-axis lie along the channel bed in the direction of the wave's propagation and take the y-axis vertically upward, with the origin lying beneath the wave crest. The shape of the free surface may be described by the expression $y = h_0[1+\zeta(x)]$, where h_0 is the depth of the fluid in the undisturbed state. Since the flow is incompressible and irrotational, there exist a harmonic velocity potential ϕ and a harmonic stream function ψ. Before proceeding with the formulation, we non-dimensionalize the problem by introducing the variables $x' = kx$, $y' = y/h_0$, $\phi' = k\phi/c$ and $\psi' = \psi/ch_0$. (For simplicity of notation, we henceforth drop the prime from each dimensionless variable.)

We find it convenient to use ϕ and ψ as independent variables, with $\psi = 0$ on the free surface, $\psi = -1$ on the channel bed and $\phi = 0$ at the wave crest. In the interior of the fluid, the continuity equation must be satisfied:

$$\kappa^2 y_{\phi\phi} + y_{\psi\psi} = 0 \quad (-1 < \psi < 0, \quad -\infty < \phi < \infty). \tag{2.1}$$

(Here $\kappa = kh_0$.) In addition, the flow must satisfy the usual boundary conditions:

$$y = 0 \quad \text{on } \psi = -1, \tag{2.2}$$

$$(y-1-\tfrac{1}{2}F^2)\,[\kappa^2(y_\phi)^2+(y_\psi)^2]+\tfrac{1}{2}F^2 = 0 \quad \text{on } \psi = 0 \tag{2.3}$$

and

$$y \to 1+\psi \quad \text{as } |\phi| \to \infty. \tag{2.4}$$

(Here $F^2 = c^2/gh_0$.) Equation (2.2) says that the channel bed is a streamline, (2.3) is Bernoulli's equation applied at the free surface (ignoring surface tension) and (2.4) says that the fluid approaches a state of uniform horizontal flow as $|\phi|$ (or $|x|$) becomes very large. We must find a function $y(\phi, \psi)$ which satisfies (2.1)–(2.4).

We begin by expanding y as a series in powers of ψ:

$$y = \sum_0^\infty a_n(\phi)\,\psi^n. \tag{2.5}$$

Since we are interested specifically in the solitary wave, we assume that

$$a_0 = 1+\sum_1^\infty A_n(\kappa^2)\,\mathrm{sech}^{2n}(\tfrac{1}{2}\phi) \tag{2.6}$$

and

$$a_1 = 1+\sum_1^\infty B_n(\kappa^2)\,\mathrm{sech}^{2n}(\tfrac{1}{2}\phi). \tag{2.7}$$

Finally, we expand A_n and B_n as series in powers of κ^2:

$$A_n = \sum_n^\infty A_{n,m}\,\kappa^{2m} \tag{2.8}$$

and

$$B_n = \sum_n^\infty B_{n,m}\,\kappa^{2m}. \tag{2.9}$$

Equation (2.1) allows us to express a_n in terms of a_0 and a_1 for $n \geqslant 2$. Equation (2.2) can then be used to express a_1 in terms of a_0. Finally, (2.3) can be used to solve for the coefficients $A_{n,m}$ in the expansion for a_0. We carried out these computations to 27th order; the results are described in the next section.

This procedure differs from I in the choice of independent variables. In I, ϕ and ψ are regarded as functions of $(x-ct)$ and y, and ϕ is expanded as a series in powers of y. The advantage of using ϕ and ψ as independent variables is that fewer computations are required; this is a consequence of the fact that we can expand about the free surface $\psi = 0$. In the formulation used in I, the position of the free surface is not known in advance.

3. Computational results and discussion

Using the coefficients $A_{n,m}$ found by the procedure outlined above, we can obtain the first 27 terms in the series expressions for the amplitude, mass and potential energy of the solitary wave. The shape of the free surface is given by

$$\begin{aligned}\zeta &= y(\phi, 0)-1\\ &= a_0(\phi)-1\\ &= \sum_1^{\infty} \kappa^{2m} \sum_1^{m} A_{n,m} \operatorname{sech}^{2n}(\tfrac{1}{2}\phi).\end{aligned} \tag{3.1}$$

From this we can derive expressions for the amplitude ϵ, mass M' and potential energy V':

$$\epsilon = \zeta(0) = \sum_1^{\infty} \kappa^{2m} \sum_1^{m} A_{n,m} = \sum_1^{\infty} \epsilon_m \kappa^{2m}, \tag{3.2}$$

$$M' = \frac{M}{h_0^2} = \kappa^{-1} \sum_1^{\infty} \mu_m \kappa^{2m} \tag{3.3}$$

and

$$V' = \frac{V}{gh_0^3} = \kappa \sum_1^{\infty} \nu_m \kappa^{2m}, \tag{3.4}$$

where

$$M = \int_{-\infty}^{\infty} h_0 \zeta(x)\,\mathrm{d}x = \frac{h_0^2}{\kappa}\int_{-\infty}^{\infty} [a_0(\phi)-1]\,\alpha_1(\phi)\,\mathrm{d}\phi \tag{3.5}$$

and

$$\begin{aligned}V &= \int_{-\infty}^{\infty}\int_{h_0}^{h_0(1+\zeta)} gy\,\mathrm{d}y\,\mathrm{d}x - Mgh_0\\ &= \frac{gh_0^3}{2\kappa}\int_{-\infty}^{\infty} [a_0(\phi)-1]^2 a_1(\phi)\,\mathrm{d}\phi.\end{aligned} \tag{3.6}$$

Computed values for the coefficients ϵ_m, μ_m and ν_m for $1 \leqslant m \leqslant 27$ are listed in table 1.

The accuracy of our computations is limited by accumulated round-off error. We use two methods for estimating this error at each order of our procedure. The first estimate is obtained by expanding the left-hand side of boundary condition (2.3), evaluated at $\phi = 0$, in powers of κ. The coefficient of κ^{2n} should be 0, so the deviation from 0 of this coefficient is a measure of the error at order n. Our second test makes use of the identity

$$3V' - (F^2-1)M' = 0 \tag{3.7}$$

(see Starr 1947). We expand the left-hand side in powers of κ^2 and observe the deviation from 0 of the coefficient of κ^{2n}. These estimates indicate that the tabulated values of ϵ_m, μ_m and ν_m are correct to the indicated degree of precision.

In I we claimed that series (3.2) gave two-digit accuracy even at $\epsilon \approx 0.52$. This claim was based on the observed convergence of the first 17 partial sums of this series. From table 1, however, we see that the coefficients ϵ_m begin to grow rapidly in magnitude for $m \geqslant 18$. As a consequence, the 18th–27th partial sums appear to converge only for $\epsilon < 0.2$. The oscillation of the signs of the coefficients ϵ_m for $m \geqslant 18$ suggests that convergence may be limited by a singularity on the negative κ^2 axis.

Following the procedure employed by Longuet-Higgins & Fenton (1974) and I, we

n	ϵ_n	μ_n	ν_n
1	0.33333333	1.33333333	0.14814815
2	0.13888889	0.44444444	0.10864198
3	0.06018519	0.15407407	0.06085832
4	0.02724758	0.04283598	0.02932236
5	0.01312908	0.00373275	0.01226941
6	0.00686689	−0.00750817	0.00413500
7	0.00394549	−0.00897396	0.00068302
8	0.00248526	−0.00759542	−0.00056168
9	0.00169132	−0.00578643	−0.00086590
10	0.00121958	−0.00424961	−0.00081953
11	0.00091544	−0.00309745	−0.00067367
12	0.00070637	−0.00227193	−0.00052354
13	0.00055481	−0.00168753	−0.00039828
14	0.00044359	−0.00127207	−0.00030171
15	0.00035082	−0.00097305	−0.00022961
16	0.00031303	−0.00075444	−0.00017628
17	0.00010763	−0.00059200	−0.00013675
18	0.00096004	−0.00046939	−0.00010721
19	−0.00496226	−0.00037554	−0.00008489
20	0.03822124	−0.00030280	−0.00006782
21	−0.31326858	−0.00024581	−0.00005461
22	2.8391377	−0.00020072	−0.00004428
23	−28.195957	−0.00016477	−0.00003613
24	305.7517	−0.00013589	−0.00002964
25	−3606.72	−0.00011253	−0.00002443
26	46120.0	−0.00009354	−0.00002022
27	−637500.0	−0.00007803	−0.00001681

TABLE 1. Coefficients in the expansions of ϵ, M' and V' in powers of κ^2

N	ϵ	M'	V'
1	0.79411765	2.7268580	0.83969144
2	0.83146580	1.9598637	0.43638013
3	0.83011677	1.9071471	0.41823134
4	0.82598975	1.8995801	0.41479180
5	0.83295182	1.8969889	0.41338448
6	0.83294811	1.8968947	0.41295379
7	0.83169582	1.8966393	0.41287509
8	0.82925479	1.8948386	0.41287471
9	0.82758678	1.8880541	0.41284289
10	0.82677737	1.8666594	0.41272344
11	0.82639193	1.8162183	0.41247453
12	0.826196	1.75748	0.412059
13	0.8261	1.735	0.4114

TABLE 2. Values of $[N,N]$ Padé approximants at $\omega = 1$

recast series (3.2)–(3.4) in terms of the parameter $\omega = 1-F^2/[y_\psi(0,0)]^2$ and use diagonal Padé approximants to accelerate convergence. The [1, 1]–[13, 13] Padé approximants for ϵ, M' and V' at $\omega = 1$ (the wave of maximum amplitude) are given in table 2. For purposes of comparison, we list in table 3 the values of ϵ, M' and V' for the highest wave as computed by Williams (1981), Witting (1981) and Hunter

	ϵ	M'	V'
Williams	0.833197	1.970319	0.437670
Witting	0.8332	1.9699 ± 0.001†	0.43758 ± 0.00022
Hunter & Vanden-Broeck	0.83322		

† This value is not given explicitly by Witting; it was obtained by means of identity (3.7).

TABLE 3. Values of ϵ, M' and V' for the highest solitary wave, as computed by Williams (1981), Witting (1981) and Hunter & Vanden-Broeck (1983)

& Vanden-Broeck (1983). In I we noted that although the approximants seemed to be converging, in agreement with Longuet-Higgins & Fenton's findings, the evidence was not entirely convincing. This statement was based only on the [1, 1]–[8, 8] approximants. From table 2 it is clear that this apparent convergence was illusory. Thus, this method may not give accurate results for waves of nearly maximum amplitude, at least not with only 27 series coefficients available.

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REFERENCES

HUNTER, J. K. & VANDEN-BROECK, J.-M. 1983 Accurate computations for steep solitary waves. *J. Fluid Mech.* **136**, 63.

LENAU, C. W. 1966 The solitary wave of maximum amplitude. *J. Fluid Mech.* **26**, 309.

LONGUET-HIGGINS, M. S. & FENTON, J. D. 1974 On the mass, momentum, energy and circulation of a solitary wave. II. *Proc. R. Soc. Lond.* A **340**, 471.

PENNELL, S. A. & SU, C. H. 1984 A seventeenth-order series expansion for the solitary wave. *J. Fluid Mech.* **149**, 431.

STARR, V. T. 1947 Momentum and energy integrals for gravity waves of finite height. *J. Mar. Res.* **6**, 175.

WILLIAMS, J. M. 1981 Limiting gravity waves in water of finite depth. *Phil. Trans. R. Soc. Lond.* A **302**, 139.

WITTING, J. 1981 High solitary waves in water: results of calculations. *NRL Rep.* 8505.